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Power Autocorrelative Function

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The spectral reflective characteristics (SRC) of the natural formations are obtained practically as a discrete sequence of values $r(\lambda_i), i=1, \ldots, n$ of the reflective index of the formation. This permits the SRC description through the vector tip $X(x_1, \ldots, x_n)$ in the finite multidimensional space with dimensionality n, in which the subject descriptive signs are $x_i \equiv r(\lambda_i)$. SRC are stochastic functions because of the natural dispersion of the formation parameters. This property of theirs necessitates the use of statistical probability methods of the SRC classification even when the measurement errors are so small as to be neglected. From the point of view of minimization of the probability R for an error of Ist gender ("omission"), or an error of 2nd gender (erroneous identification "false alarm"), most suitable proves to be the Bayes method of the minimum risk at the SRC classification analysis. In the general case, when the autocorrelative SRC matrix is not diagonal, the analytical conclusions become very difficult. But in some particular cases it is possible to formulate relatively simple criteria of the risk function R magnitude at the determination appurtenance of the vector-observation $X(x_1, \ldots, x_n)$ to one of the two classes — k or m. For instance, the risk function simplifies considerably at the following limiting conditions:

a) Signs x are independent (particularly $n=1$),

b) SRC of the two vector-realizations are connected with the relation: $r_m(\lambda)=(1+\Theta)r_k(\lambda)$, $\Theta=\text{const}\ll 1$.

c) k- and m-classes have normal distributions with parameters $\mu_{k_i}, \mu_{m_i} = (1+\Theta)\mu_{k_i}$, $\sigma_{k_i}=V\mu_{k_i}$, $\sigma_{m_i}=V\mu_{k_i}=V(1+\Theta)\mu_{k_i}$, where μ_{k_i} is the mathematical expectation of $r_k(\lambda_i)$, and σ_{k_i} is the $r_k(\lambda_i)$ variance. The constant V is the variational index. As shown in [3], under these conditions the risk function R is measured with a normalized normal distribution having a coordinate equal approximately to

$$\eta=\Theta/V \tag{1}$$

(after neglecting the high powers of Θ).

As in that case, R is determined by the integral $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\eta^2/2} dy, \ \eta<0,$ then the value of R is smaller at a great absolute value of η.

Of course, these limiting conditions are quite strong (condition (b) in particular), but they provide a possibility for analytical conclusions through the criterion (1). They can serve as an orientation for the situation in the general case of the SRC nondiagonal covariance matrix.

The space of $X(x_1, \ldots, x_n)$ can be formed not only through $r(\lambda)$, but also by the $r(\lambda)$ transformations with a suitable operator. Such a transformation has justification of performance only when the risk function value could be reduced in the transformed space. Paper [3] shows that the autocorrelative transformations:

$$(2) \quad C_M(\tau_j) = \sum_{i=1}^{n} | r(\lambda_i) - r(\lambda_i + \tau_j) |, \ \tau_j = \Delta\lambda . j, \ j = 1, \ldots, n/2, \ \Delta\lambda = \lambda_{i+1} - \lambda_i = \text{const},$$

$$(3) \qquad C_k(\tau_j) = \sum_{i=1}^{n} [r(\lambda_i) - r(\lambda_i + \tau_j)]^2,$$

improve the risk function for the cases described by the limiting conditions (a), (b) and (c) with the increase of the $|\eta|$ value. The transformation (3) is given by Kolmogorov [1] and the transformation (2) is defined in [2].

This paper examines the generalization of (2) and (3), namely:

$$(4) \qquad C(\tau_j) = \sum_{i=1}^{n} [| r(\lambda_i) - r(\lambda_i + \tau_j) |]^N, \quad N = 1, \ldots, \infty.$$

We shall accept equation (1) as an effectivity criterion of this transformation. The studies in paper [3] show that C_k from equation (3) leads to a smaller value of $|\eta|$ than C_M from equation (2). In the general case this justifies the examination of the ratio

$$(5) \qquad Q = \frac{\eta_{N-m}}{\eta_N} = \frac{\Theta_{N-m} V_N}{\Theta_N V_{N-m}}.$$

Taking into consideration the limiting condition (b) and equation (4), we obtain the small parameter Q_n by the relation

$$(6) \qquad C_m^{(N)}(\tau_j) = (1+\Theta)^N C_K^{(N)}(\tau_j).$$

If we neglect the high powers of Θ in (6), we would obtain $\Theta_N = N\Theta$ and then

$$(7) \qquad \Theta_N / \Theta_{N-m} = N/(N-m).$$

In order to study the V_{N-m}/V_N ratio it is necessary to determine the expressions for μ_{C_N} and σ_{C_N} of $C^{(N)}$ for an arbitrary N. This can be realized by the use of the definition equation (4) and of the dispersion equation of a normal distribution composition.

(8) $$y=\sum_i a_i x_i,$$

$$\sigma_y^2=\sum_i a_i^2\sigma_{x_i}^2 .$$

In equation (4) the lth realization $r_l(\lambda_i)$ of a given class can be expressed as follows:

(9) $$r_l(\lambda_i)=\overline{r(\lambda_i)}+\Delta r_l(\lambda_i),$$

where $\overline{r(\lambda_i)}$ is the mathematical expectation of r in λ_i for the examined class. $r_l(\lambda_i+\tau_j)$ is expressed analogously:

(9a) $$r_l(\lambda_i+\tau_j)=\overline{r(\lambda_i+\tau_j)}+\Delta r_l(\lambda_i+\tau_j).$$

Taking into consideration that there follows from condition b) that $V\ll 1$, the high powers of Δr_l can be neglected and then we obtain after the substitution of (9) and (9a) in (4) in a first approximation:

(10) $$C_l^{(N)}\approx\sum_{i=1}^{n}[(\bar{x}_i-\bar{y}_i)^N+N(\bar{x}_i-\bar{y}_i)^{N-1}(\Delta x_{i_l}-\Delta y_{i_l})],$$

where it is marked for convenience: $x_i=r(\lambda_i)$, $y_i=r(\lambda_i+\tau_j)$.

It follows from equation (8) and (10) and condition b) that:

(11) $$\overline{C^{(N)}}=\sum_{i=1}^{n}(\bar{x}_i-\bar{y}_i)^N,$$

(12) $$\sigma_{C_N}^2=V^2N^2\sum_{i=1}^{n}(\bar{x}_i-\bar{y}_i)^{2(N-1)}(\bar{x}_i^2+\bar{y}_i^2).$$

The ratio V_N/V_{N-m} is expressed by equations (11) and (12) in the following way:

(13) $$\frac{V_N}{V_{N-m}}=\frac{N}{N-m}\sqrt{\frac{\left[\sum_{i=1}^{n}(\bar{x}_i-\bar{y}_i)^{N-m}\right]^2\left[\sum_{i=1}^{n}(\bar{x}_i-\bar{y}_i)^{2(N-1)}(\bar{x}_i^2+\bar{y}_i^2)\right]}{\left[\sum_{i=1}^{n}(\bar{x}_i-\bar{y}_i)^{2(N-m-1)}(\bar{x}_i^2+\bar{y}_i^2)\right]\left[\sum_{i=1}^{n}(\bar{x}_i-\bar{y}_i)^N\right]^2}}.$$

There follows from (5), (7) and (13) that the ratio (5) has the form

(14) $$\Omega=\sqrt{\frac{\left[\sum_{i=1}^{n}(\bar{x}_i-\bar{y}_i)^{N-m}\right]^2\left[\sum_{i=1}^{n}(\bar{x}_i-y_i)^{2(N-1)}(x_i^2+\bar{y}_i^2)\right]}{\left[\sum_{i=1}^{n}(x_i-\bar{y}_i)^{2(N-m-1)}(\bar{x}_i^2+\bar{y}_i^2)\right]\left[\sum_{i=1}^{n}(x_i-\bar{y}_i)^N\right]^2}},$$

or it can be written by triple indices:

$$\text{(14a)} \qquad \Omega=\frac{\sum_{i,j,k}(a_i^{N-m}a_j^{N-m}a_k^{2(N-1)}b_k)}{\sum_{i,j,k}(a_i^N a_j^N a_k^{2(N-m-1)}b_k)}, \quad \begin{array}{l} i=1,\ldots,n, \\ j=1,\ldots,n, \\ k=1,\ldots,n, \end{array}$$

$$a_i=|\bar{x}_i-\bar{y}_i|, \quad b_k=\bar{x}_k^2+\bar{y}_k^2.$$

In order to evaluate the effectivity of the N increase, it is necessary to examine the extremums of Ω with respect to N (for that purpose equation (14a) is suitable), and the extremum of the structure of $r(\lambda)$ (equation (14) suits the purpose). In particular, the following system should be solved:

$$\text{(15a)} \qquad \partial\Omega/\partial x_i=0,$$

$$\text{(15b)} \qquad \partial\Omega/\partial y_i=0, \quad i=1,\ldots,n,$$

$$\text{(15c)} \qquad \partial\Omega/\partial N=0.$$

There follows from (14) that (15a) and (15b) contain x_i and y_i in a symmetric manner, i. e. the results from (15a) will be valuable also for (15b). We obtain for (15a) in a developed form:

$$\text{(16a)} \quad \frac{1}{2}\frac{\partial\Omega}{\partial x_i}=\{\sqrt{A}B(N-m)(\bar{x}_i-\bar{y}_i)^{N-m-1}+A[(N-1)(x_i-\bar{y}_i)^{2N-3}(\bar{x}_i^2+\bar{y}_i^2)$$
$$+(\bar{x}_i-\bar{y}_i)^{2(N-1)}x_i]\}CD-\{C\sqrt{D}N(\bar{x}_i-\bar{y}_i)^{N-1}$$
$$+D[(N-m-1)(\bar{x}_i-\bar{y}_i)^{2(N-m)-3}\cdot(\bar{x}_i^2+\bar{y}_i^2)+(x_i-y_i)^{2(N-m-1)}\cdot\bar{x}_i]\}AB,$$

where

$$A=\left[\sum_{i=1}^{n}(\bar{x}_i-\bar{y}_i)^{N-m}\right]^2, \quad C=\sum_{i=1}^{n}(x_i-\bar{y}_i)^{2(N-m-1)}\cdot(\bar{x}_i^2+\bar{y}_i^2),$$

$$B=\sum_{i=1}^{n}(\bar{x}_i-\bar{y}_i)^{2(N-1)}\cdot(\bar{x}_i^2+\bar{y}_i^2), \quad D=\left[\sum_{i=1}^{n}(x_i-\bar{y}_i)^N\right]^2.$$

The sums A, B, C, D in (16a) are independent of the index i, and $\bar{x}_i$ as well as $\bar{y}_i$ takes part in the remaining part of the equation in an equal manner for the different values of the index i. Therefore, the system (16a) is reduced to a single equation, representing a polynomial of x. Still, if we take the difference x_i-y_i by module (according to the definition equation (4)), then $\bar{y}_i$ participates symmetrically to x for each i and in all equations. That is why, the simultaneous satisfaction of (16a) and (16b) demands:

$$\text{(17)} \qquad \bar{x}_i=\bar{y}_i.$$

But it follows from the definition equation (4) that if (17) is fulfilled then $x=\text{const}$.

The derivative (15) is expressed in a developed form by the equation

$$\text{(16b)} \quad \frac{\partial \Omega}{\partial N} = \frac{\sum\limits_{i,j,k,p,q,r} \left[a_i^{N-m} a_j^{N-m} a_k^{2(N-1)} a_p^N a_q^N a_r^{2(N-m-1)} b_k^2 l_n \frac{a_i a_j a_k^2}{a_p a_q a_r^2} \right]}{\left[\sum\limits_{i,j,k} a_i^N a_j^N a_k^{2(N-m-1)} b_k \right]^2} = 0.$$

$(i, j, k, p, q, r) = 1, \ldots, n.$

Obviously (16b) is annulled also by the condition $x = \text{const}$. Therefore, Ω from (14a) has an extremum and it appears when $r(\lambda) = \text{const}$.

At $N \to \infty$ the sequence $\{(\bar{x}_i - \bar{y}_i)\}$, $i = 1, \ldots, n$ tends to the sequence $\{0, 0, \ldots, 0, (\bar{x}_j - \bar{y}_j)_{\max}\}$, $j = 1, \ldots, K$, where K is the number of the biggest and equal in size differences $(\bar{x}_j - \bar{y}_j)_{\max}$. That is why the conditions for an extremum of (14a) are realized at $N \to \infty$.

A direct verification can prove that the extremum defined by the system (15a), (15b) and (15c) is a maximum.

Conclusions

According to the results obtained, each power autocorrelative function, defined by equation (4), diminishes the risk function of equation (1) when its power index increases. But we should not forget that these results have been obtained under the following limiting conditions: $V \ll 1$, $r_k(\lambda) = (1+\Theta) r_m(\lambda)$, $\Theta \ll 1$. The increase of the power index N leads to an increase of the role of the neglected terms in the development of μ_{C_N}, σ_{C_N} and $(1+\Theta)^N$; for example, at $N > 10$ their contribution in some cases could be higher than 30—40 per cent even when $V = 0.05$ and $\Theta = 0.05$. Nevertheless, at relatively small values of N, it is possible to look for an optimum of each concrete set of classes. It is probable that this optimum would be shifted towards the great values of N, when for each one of the M-classes there exists at least one wavelength λ, in which this class has the highest values of $r(\lambda)$, compared with the other classes.

If we take into consideration the higher powers of Δx and Δy in the expression for $C_i^{(N)}$ in equation (10), then the distribution of $C_i^{(N)}$ would not be a composition of normal distribution, therefore equations (11) and (12) would not be valuable as sufficient parameters of that distribution description. This would complicate considerably the analytical conclusions for the effectivity of transformations (4).

Notwithstanding the fact that the analytical conclusions of this paper are quite limited, they do provide grounds to expect good effectivity of transformation (4). The verification of this effectivity in the general case of multidimensional distributions should be realized as the file of primary information for $r(\lambda)$ has to be transformed into a file of power autocorrelative functions, according to (4), and the Bayes procedure of the minimum risk or some other convenient criteria should be applied to this file.

References

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Степенная аутокорреляционная функция

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(Резюме)

Исследованы свойства аутокорреляционной функции вида

$$C(\tau_j)=\sum_{i=1}^{n}[r(\lambda_i)-r(\lambda_i+\tau_j)]^m,$$

где m — целое положительное число, τ_j — шаг корреляции, λ_i — независимая переменная, $r(\lambda_i)$ — зависимая переменная.

Показано, что применение $C(\tau_j)$ в классификационном анализе стохастических сигналов типа $r(\lambda_i)$ уменьшает функцию риска при нормальном распределении $r(\lambda_i)$.